UNIVERSITY OF WATERLOO
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ECE 204 *Numerical methods*

Approximating solutions to systems of higher-order initial-value problems

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Introduction

- In this topic, we will
  - Discuss converting systems of higher-order initial-value problems into a system of 1^{st}-order initial-value problems
  - Look at an example

System of two 2nd-order initial-value problems

- We will show this by example:

$$y^{(2)}(t) + 2y^{(1)}(t) + y(t) + z(t) = \sin(t)$$
$$z^{(2)}(t) + z^{(1)}(t) + z(t) + y(t) = \cos(t)$$

  – This requires four initial conditions:

$$y(t_0) = y_0$$
$$y^{(1)}(t_0) = y_0^{(1)}$$
$$z(t_0) = z_0$$
$$z^{(1)}(t_0) = z_0^{(1)}$$

System of two 2nd-order initial-value problems

- We will represent:

$$y(t) = w_0(t)$$
$$y^{(1)}(t) = w_1(t)$$
$$z(t) = w_2(t)$$
$$z^{(1)}(t) = w_3(t)$$

  – We can immediately translate the initial conditions:

$$y(t_0) = y_0 \qquad w_0(t_0) = y_0$$
$$y^{(1)}(t_0) = y_0^{(1)} \qquad w_1(t_0) = y_0^{(1)}$$
$$z(t_0) = z_0 \qquad w_2(t_0) = z_0$$
$$z^{(1)}(t_0) = z_0^{(1)} \qquad w_3(t_0) = z_0^{(1)}$$

System of two 2nd-order initial-value problems

- We also immediately note that:

$$w_0^{(1)}(t) = y^{(1)}(t) = w_1(t)$$
$$w_2^{(1)}(t) = z^{(1)}(t) = w_3(t)$$

- That leaves us with determining $w_1^{(1)}(t), w_3^{(1)}(t)$

$$y^{(2)}(t) = \sin(t) - 2y^{(1)}(t) - y(t) - z(t)$$
$$z^{(2)}(t) = \cos(t) - z^{(1)}(t) - z(t) - y(t)$$

$$w_1^{(1)}(t) = \sin(t) - 2w_1(t) - w_0(t) - w_2(t)$$
$$w_3^{(1)}(t) = \cos(t) - w_3(t) - w_2(t) - w_0(t)$$

System of two 2nd-order initial-value problems

- Thus, we have:

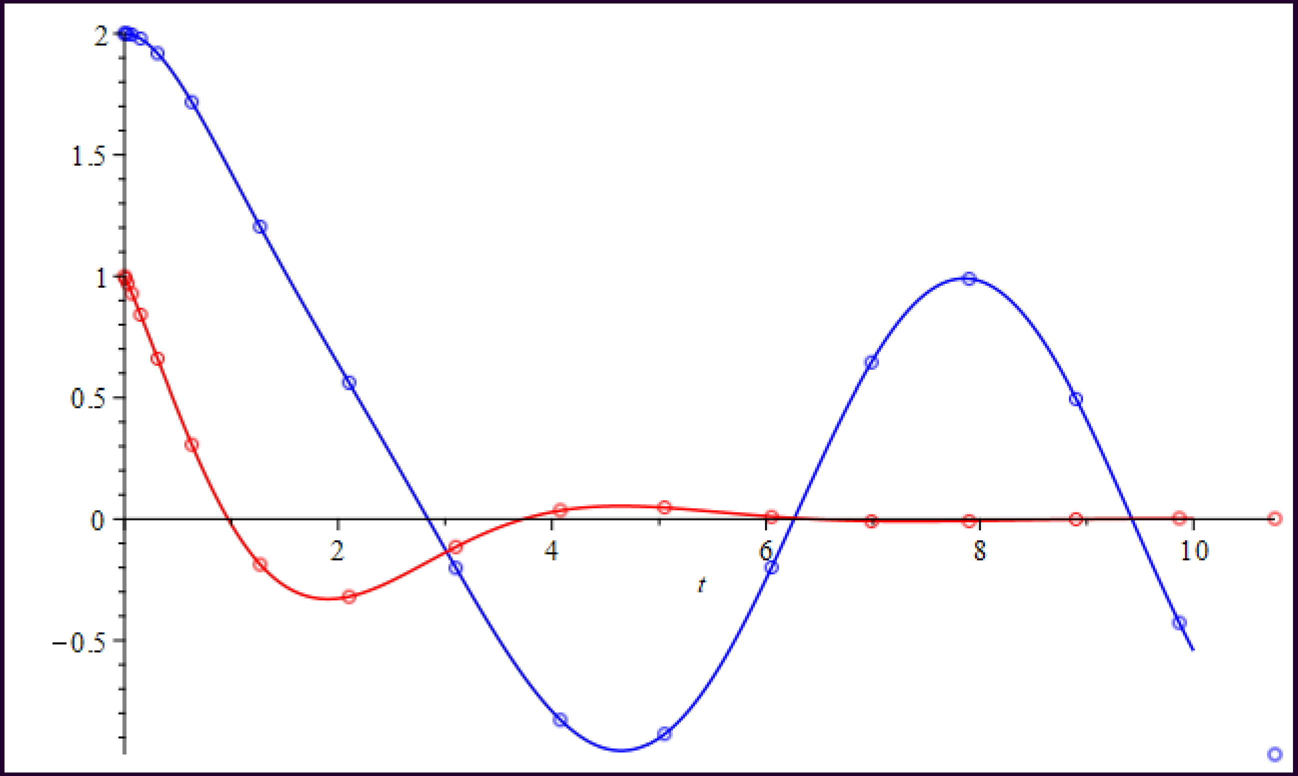
$$\mathbf{w}^{(1)}(t) = \begin{pmatrix} w_1(t) \\ \sin(t) - 2w_1(t) - w_0(t) - w_2(t) \\ w_3(t) \\ \cos(t) - w_3(t) - w_2(t) - w_0(t) \end{pmatrix} \qquad \mathbf{w}(t_0) = \begin{pmatrix} y_0 \\ y_0^{(1)} \\ z_0 \\ z_0^{(1)} \end{pmatrix}$$

- We can write this as a function:

```
vec<4> f( double t, vec<4> w ) {
    return vec<4>{
        w[1],
        std::sin(t) - 2.0*w[1] - w[0] - w[2],
        w[3],
        std::cos(t) -     w[3] - w[2] - w[0]
    };
}
```

System of two 2^{nd}-order initial-value problems

- Plotting the actual solution versus the



System of two 2nd-order initial-value problems

$w_0(t) = y(t) \quad w_1(t) = y^{(1)}(t) \quad w_2(t) = z(t) \quad w_3(t) = z^{(1)}(t)$

```
     0           < 1           -1           2           0        >
     0.01        < 0.9899507   -1.0098008   1.9999005  -0.0198502>
     0.03        < 0.9695678   -1.0282224   1.9991135  -0.0586547>
     0.07        < 0.9277737   -1.0604838   1.9952704  -0.1327130>
     0.15        < 0.8408958   -1.1077690   1.9791628  -0.2669310>
     0.31        < 0.6599369   -1.1418081   1.9182831  -0.4828253>
     0.63        < 0.3072558   -1.0298129   1.7176558  -0.7357848>
     1.27        <-0.1870949   -0.4857930   1.2033141  -0.8119151>
     2.1015668   <-0.3197583    0.0966596   0.5615785  -0.7482154>
18   3.1015668   <-0.1142981    0.2232183  -0.2015183  -0.7630420>
     4.0791478   < 0.0363239    0.0711120  -0.8282826  -0.4258318>
     5.0504978   < 0.0478470   -0.0300383  -0.8858265   0.3398003>
     6.0504978   < 0.0094451   -0.0317667  -0.1990892   0.9286665>
     6.9858539   <-0.0096116   -0.0052132   0.6446146   0.7403237>
     7.9002755   <-0.0081134    0.0075244   0.9883434  -0.0443953>
     8.9002755   <-0.0001790    0.0047968   0.4956264  -0.8559776>
     9.8671404   < 0.0035653   -0.0015659  -0.4269945  -0.8992590>
     10.762308   < 0.0023111   -0.0031120  -0.9703157  -0.2322476>
```

Other systems higher-order initial-value problems

- Thus, if we had four coupled ODEs:

$$u_1^{(4)}(t) = f_1\left(t, u_1(t), \ldots, u_4^{(2)}(t)\right)$$
$$u_2^{(2)}(t) = f_2\left(t, u_1(t), \ldots, u_4^{(2)}(t)\right)$$
$$u_3^{(2)}(t) = f_3\left(t, u_1(t), \ldots, u_4^{(2)}(t)\right)$$
$$u_4^{(3)}(t) = f_4\left(t, u_1(t), \ldots, u_4^{(2)}(t)\right)$$

  - This requires eleven initial conditions
  - This would require us to define a system of eleven 1st-order initial-value problems

Other systems higher-order initial-value problems

- Thus we would reformulate as follows:

$$u_1^{(4)}(t) = f_1\left(t, u_1(t), \ldots, u_4^{(2)}(t)\right)$$

$$u_2^{(2)}(t) = f_2\left(t, u_1(t), \ldots, u_4^{(2)}(t)\right)$$

$$u_3^{(2)}(t) = f_3\left(t, u_1(t), \ldots, u_4^{(2)}(t)\right)$$

$$u_4^{(3)}(t) = f_4\left(t, u_1(t), \ldots, u_4^{(2)}(t)\right)$$

$$w_0^{(1)}(t) = w_1(t)$$

$$w_1^{(1)}(t) = w_2(t)$$

$$w_2^{(1)}(t) = w_3(t)$$

$$w_3^{(1)}(t) = f_1(t, \mathbf{w}(t))$$

$$w_4^{(1)}(t) = w_5(t)$$

$$w_5^{(1)}(t) = f_2(t, \mathbf{w}(t))$$

$$w_6^{(1)}(t) = w_7(t)$$

$$w_7^{(1)}(t) = f_3(t, \mathbf{w}(t))$$

$$w_8^{(1)}(t) = w_9(t)$$

$$w_9^{(1)}(t) = w_{10}(t)$$

$$w_{10}^{(1)}(t) = f_4(t, \mathbf{w}(t))$$

Summary

- Following this topic, you now
  - Understand how to convert a system of higher-order initial-value problems into a system of 1st-order initial-value problems
  - Have seen an example and its solution

References

[1] https://en.wikipedia.org/wiki/Initial_value_problem

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.

Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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